Assessing Interactive Causes of an Occurred Outcome Due to Two Binary Exposures

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Causal identification and discovery (CIFW02)

Outline

Introduction

The Three Levels of the "Ladder of Causation"

- **Level 3: Counterfactuals**

  Today's focus

  ▷ Legal liability allocation

  ▷ Medical compensation

- **Level 2: Intervention**

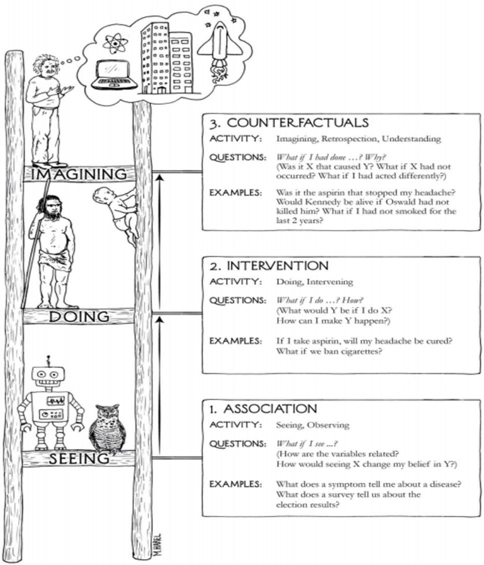
  ▷ Treatment effect evaluation

  ▷ Public health policy design

- **Level 1: Association**

  ▷ Risk prediction

  ▷ Correlation



Effects of Causes vs. Causes of Effects

Level 2 (Intervention, Effects of Causes)

- Prospective, forward-looking, make *predictions* of some intervention
- Question: "If we ban smoking, how much will lung cancer rates decrease?"
- Mathematics: average treatment effect, conditional average treatment effect
- Well-developed: propensity scores, DR, IV, proximal inference

Level 3 (Counterfactuals, Causes of Effects)

- Retrospective, backward-looking, makes *inferences* about what already happened
- Question: "Given that this worker smoked and developed lung cancer — would he still have developed lung cancer if he had never smoked?"
- Mathematics: Condition on *observed* outcome, ask about the true cause
- Application: legal liability, environmental litigation, insurance compensation

Literature: Single Binary Cause I

Probability of Necessity (Pearl, 2000; Tian & Pearl, 2000)

$$\textbf{PN} = \text{pr}(Y_{Z=0} = 0 \mid Z = 1, Y = 1)$$

- Conditioning on the observed exposure and outcome: PN gives the probability that exposure was *necessary* for the occurred outcome
- $\mathrm{PN} = 1$ means the exposure was an absolutely necessary condition for the outcome to occur. Without the exposure, the outcome would not have occurred.
- **Key Challenge:** Randomization is the gold standard of causal inference and identifies interventional quantities such as $\mathbb{E}(Y_{Z=1} - Y_{Z=0})$ (Level 2), but it cannot identify PN (Level 3).

Pearl, Judea (2000). *Causality: Models, Reasoning, and Inference.* Cambridge University Press.

Tian, Jin, and Judea Pearl (2000). Probabilities of causation: Bounds and identification. *Annals of Mathematics and Artificial Intelligence*, **28**(1), 287-313.

Literature: Single Binary Cause II

Probability of Causation (Dawid 2000; Dawid & Musio, 2022)

$$\mathrm{PC} = \mathrm{pr}(Y_{Z=0} = 0,\ Y_{Z=1} = 1 \mid Z = 1, Y = 1)$$

- Conditioning on the observed exposure and outcome: PC gives the probability that the exposure was both necessary for the outcome (wouldn't have occurred without it) and sufficient (caused it to occur)
- Given the consistency assumption, PC is equivalent to PN
- **Identification:** Under randomization + monotonicity ($Y_{Z=1} \geq Y_{Z=0}$):

$$\mathrm{PN} = \mathrm{PC} = 1 - \mathrm{pr}(Y = 1 \mid Z = 0)/\mathrm{pr}(Y = 1 \mid Z = 1) = 1 - RR.$$

Dawid, A. Philip (2000). Causal inference without counterfactuals. *Journal of the American Statistical Association*, **95**(450), 407-424.

Dawid, A. Philip, and Monica Musio (2022). Effects of causes and causes of effects. *Annual Review of Statistics and Its Application*, **9**, 261-287.

Literature: Single Binary Cause III

Extension to Ordinal Outcomes (Zhang et al., 2025)

Generalized PN for a binary cause and an ordinal outcome $Y \in \{0, 1, \dots, J-1\}$:

$$\mathrm{PN}(\omega_0, y) = \mathrm{pr}(Y_{Z=0} < y \mid Z = 1, Y = y)$$

- Conditioning on observed exposure and outcome ($Z = 1$, $Y = y$): this probability quantifies the likelihood that the outcome would have been strictly lower had the exposure been absent.
- **Identification:** Under randomization and monotonicity with individual treatment effects bounded by 1

Zhang, Chao, Zhi Geng, Wei Li, and Peng Ding. (2025). Identifying and bounding the probability of necessity for causes of effects with ordinal outcomes. *Biometrika*, in press.

Literature: Multiple Causes

Posterior Total Causal Effect (Lu et al., 2023; Li et al., 2024)

Setting: p binary causes $X_1, \ldots, X_p$, binary outcome Y (multiple outcomes)

$$\mathrm{PostTCE}(X_j \Rightarrow Y) = \mathbb{E}(Y_{X_j=1} - Y_{X_j=0} \mid X_1 = x_1, \ldots, X_p = x_p,\ Y = 1)$$

- Quantifies the average contribution of each cause X_j separately, conditional on the observed binary causes and binary outcome(s)
- Identification is achieved under *sequential ignorability* and *sequential monotonicity* assumptions
- When $p = 1$ (single cause): $\mathrm{PostTCE}(X_1 \Rightarrow Y)$ reduces to PN

Lu, Zitong, Zhi Geng, Wei Li, Shengyu Zhu, and Jinzhu Jia. (2023). Evaluating causes of effects by posterior effects of causes. *Biometrika*, **110**(2), 449-465.

Li, Wei, Zitong Lu, Jinzhu Jia, Min Xie, and Zhi Geng (2024). Retrospective causal inference with multiple effect variables. *Biometrika*, **111**(2), 573-589.

Legal Liability with a Single Cause

Scenario: A lung cancer patient who smoked, i.e., $\mathcal{O}_1 = (Z = 1, Y = 1)$.

Goal: Quantify the responsibility attributed to the tobacco company.

	Causal Explanation	**Smoking**
$\text{pr}(Y_{Z=0} = 0, Y_{Z=1} = 1 \mid Z = 1, Y = 1)$	Smoking is the cause	100%
$\text{pr}(Y_{Z=0} = 1, Y_{Z=1} = 1 \mid Z = 1, Y = 1)$	Doomed (always occurs)	0%

Smoking Responsibility: $\mathrm{PN} = \mathrm{PC} = \text{pr}(Y_{Z=0} = 0, Y_{Z=1} = 1 \mid Z = 1, Y = 1)$.

Legal Interpretation

The tobacco company bears responsibility proportional to PC; the remaining $1 - \mathrm{PC}$ reflects cases where cancer would have occurred regardless of smoking (with responsibility falling on the patient themselves or other factors).

PN or PC is sufficient for addressing the question of attributing responsibility for a cause of effect.

Revisiting PN for two causes I

Smoking, Asbestos, and Lung Cancer

Based on VanderWeele (2014) & Hilt et al. (1986), we consider lung cancer rates among 21,319 Norwegian males.

Notation: $Z = 1$ (smoking), $M = 1$ (asbestos), $Y = 1$ (lung cancer)

$$\text{pr}(Y = 1 \mid Z = 0, M = 0) = 0.12\%$$

$$\text{pr}(Y = 1 \mid Z = 0, M = 1) = 0.67\%$$

$$\text{pr}(Y = 1 \mid Z = 1, M = 0) = 0.95\%$$

$$\text{pr}(Y = 1 \mid Z = 1, M = 1) = 4.51\%$$

Lung cancer rates vary substantially across exposure groups.

Dual exposure yields a substantially high cancer rate!

A typical case:

- Some workers with asbestos exposure ($M = 1$)
- With a history of smoking ($Z = 1$)
- Diagnosed with lung cancer ($Y = 1$)
- Now in court — **who pays?**

Two legal questions

- Did smoking, asbestos, or their *interaction* cause the cancer?
- How much should the tobacco company and the asbestos company each pay?

Revisiting PN for two causes II

Both PN and PostTCE measure the total effect of each cause given observed evidence, but neither captures the *synergistic mechanism* by which two causes act jointly to produce the outcome.

Example: Smoking and Asbestos

- Consider applying single-cause attribution methods to analyze each exposure separately.
- Suppose PN estimation yields $\mathrm{PN}(\text{smoking}) = \text{pr}(Y_{Z=0} = 0 \mid Z = 1, Y = 1) = 0.6$ and $\mathrm{PN}(\text{asbestos}) = \text{pr}(Y_{M=0} = 0 \mid M = 1, Y = 1) = 0.5$.
- **Causal mechanism**: single-cause measures cannot capture the synergistic contribution of smoking and asbestos.
- **Responsibility allocation**: a naive normalization gives

$$\frac{0.6}{1.1} \approx 54.5\% \quad \text{vs.} \quad \frac{0.5}{1.1} \approx 45.5\%$$

$\Rightarrow$ PN values do not sum to 1, are not proportions, and provide no principled basis for apportioning liability between two causes.

Outline

Attribution Framework

Setup & Notation I

Two binary exposures/causes, one binary outcome:

- Z: Smoking ($Z = 1$ yes, $Z = 0$ no)
- M: Asbestos exposure ($M = 1$ yes, $M = 0$ no)
- Y: Lung cancer ($Y = 1$ yes, $Y = 0$ no)

Potential outcomes: $Y_{z,m}$ = outcome this person *would have* under $Z = z, M = m$

- $Y_{0,0}$: Potential outcome under neither exposure (baseline)
- $Y_{0,1}$: Potential outcome under asbestos only
- $Y_{1,0}$: Potential outcome under smoking only
- $Y_{1,1}$: Potential outcome under both exposures

Key point: Every individual has these four potential outcomes, but we only *observe one* – the one corresponding to their actual exposures.

Consistency: $Y = Y_{Z,M}$ (observed outcome = potential outcome under actual exposures)

Each individual's *causal type* $G = rstu$ encodes their potential outcomes under all four exposure combinations, where $r, s, t, u \in \{0, 1\}$:

$$G = (Y_{Z=0,M=0},\ Y_{Z=0,M=1},\ Y_{Z=1,M=0},\ Y_{Z=1,M=1}) = (r,\ s,\ t,\ u)$$

Its population probability is:

$$\pi_{rstu} = \mathrm{pr}(Y_{Z=0,M=0} = r,\ Y_{Z=0,M=1} = s,\ Y_{Z=1,M=0} = t,\ Y_{Z=1,M=1} = u)$$

Key Examples:

- $G = 0001$: Cancer occurs *only if* both smoking and asbestos are present (synergistic)
- $G = 0011$: Cancer occurs *only if* smoking is present (Z sufficient)
- $G = 0101$: Cancer occurs *only if* asbestos is present (M sufficient)
- $G = 0111$: Either smoking or asbestos alone causes cancer (either sufficient)
- $G = 1111$: Cancer occurs regardless of exposures (doomed)

Type	$(Y_{0,0}, Y_{0,1}, Y_{1,0}, Y_{1,1})$	Class Description
1	(0,0,0,0)	Lung cancer never occurs (immune).
2	(0,0,0,1)	Lung cancer occurs if and only if smoking with asbestos exposure is present (synergistic).
3*	(0,0,1,0)	Lung cancer occurs if and only if smoking without asbestos exposure is present.
4	(0,0,1,1)	Lung cancer occurs if and only if smoking is present (smoking).
5*	(0,1,0,0)	Lung cancer occurs if and only if non-smoking with asbestos exposure is present.
6	(0,1,0,1)	Lung cancer occurs if and only if asbestos exposure is present (asbestos).
7*	(0,1,1,0)	Lung cancer occurs if and only if either non-smoking with asbestos exposure, or smoking without asbestos exposure, is present.
8	(0,1,1,1)	Lung cancer occurs if and only if either smoking or asbestos exposure is present (parallel).
9*	(1,0,0,0)	Lung cancer disappears if and only if either smoking or asbestos exposure is present.
10*	(1,0,0,1)	Lung cancer disappears if and only if either non-smoking with asbestos exposure, or smoking without asbestos exposure, is present.
11*	(1,0,1,0)	Lung cancer disappears if and only if asbestos exposure is present.
12*	(1,0,1,1)	Lung cancer disappears if and only if non-smoking with asbestos exposure is present.
13*	(1,1,0,0)	Lung cancer disappears if and only if smoking is present.
14*	(1,1,0,1)	Lung cancer disappears if and only if smoking without asbestos exposure is present.
15*	(1,1,1,0)	Lung cancer disappears if and only if smoking with asbestos exposure is present.
16	(1,1,1,1)	Lung cancer always occurs (doomed).

The red types remain valid under the monotonicity assumption $Y_{0,0} \le Y_{0,1} \le Y_{1,1}$ and $Y_{0,0} \le Y_{1,0} \le Y_{1,1}$.

Posterior Probabilities of Causal Types

The posterior probability of latent class $G = rstu$ given observed evidence $\mathcal{O}_{z,m} = (Z = z, M = m, Y = 1)$ is

$$\pi_{rstu}(\mathcal{O}_{z,m}) = \text{pr}\left(\begin{matrix} Y_{Z=0,M=0} = r, Y_{Z=0,M=1} = s, \\ Y_{Z=1,M=0} = t, Y_{Z=1,M=1} = u \end{matrix} \;\middle|\; \mathcal{O}_{z,m} \right)$$

- Posterior probability is a generalization of PN and PC for two binary causes.
- If $\mathcal{O}_{z,m} = \mathcal{O}_{1,1}$, then $\pi_{rstu}(\mathcal{O}_{1,1})$ captures the likely cause(s) of observed lung cancer in those exposed to both smoking and asbestos (evidence).
- $\pi_{0001}(\mathcal{O}_{1,1}) = 1$ implies: among those who smoked, were exposed to asbestos, and developed lung cancer, the synergistic effect of both exposures ($Z = 1$ & $M = 1$) is the true cause.
- $\pi_{0011}(\mathcal{O}_{1,1}) = 1$ implies: among those who smoked, were exposed to asbestos, and developed lung cancer, smoking alone ($Z = 1$) is the true cause.

Why Do Posterior Probabilities Matter? - Causal Mechanism

The court asks: For this patient's cancer, which cause was more important?

We Can Calculate the Probability of Each Mechanism

For evidence $\mathcal{O}_{1,1} = (Z = 1, M = 1, Y = 1)$, posterior type probabilities:

- **Synergistic:** **71.01%**

  $\text{pr}(G = 0001 \mid \mathcal{O}_{1,1}) = 71.01\%$

  Both causes essential

- **Smoking-only:** 14.16%

  $\text{pr}(G = 0011 \mid \mathcal{O}_{1,1}) = 14.16\%$

  Smoking key, asbestos irrelevant

- **Asbestos-only:** 7.84%

  $\text{pr}(G = 0101 \mid \mathcal{O}_{1,1}) = 7.84\%$

  Asbestos key, smoking irrelevant

- **Parallel:** 4.35%

  $\text{pr}(G = 0111 \mid \mathcal{O}_{1,1}) = 4.35\%$

  Either alone sufficient

- **Doomed:** 2.64%

  $\text{pr}(G = 1111 \mid \mathcal{O}_{1,1}) = 2.64\%$

  Neither cause mattered

- **other 11 types:** 0.00%

  $\text{pr}(G = rstu \mid \mathcal{O}_{1,1}) = 0.00\%$

  ...

The synergistic interaction between smoking and asbestos is the main cause!

Why Do Posterior Probabilities Matter? - Legal Responsibility

Scenario: A lung cancer patient with both smoking and asbestos exposure, i.e., $\mathcal{O}_{1,1} = (Z = 1, M = 1, Y = 1)$.

Goal: Quantify the proportion of responsibility attributed to the tobacco and asbestos companies.

Posterior Probabilities $\pi_{rstu}(\mathcal{O}_{1,1})$ $\text{pr}(Y_{0,0} = r, Y_{0,1} = s, Y_{1,0} = t, Y_{1,1} = u \mid \mathcal{O}_{1,1})$	**Causal Explanation**	**Smoking**	**Asbestos**
$\pi_{0001}(\mathcal{O}_{1,1}) = \text{pr}(G = 0001 \mid \mathcal{O}_{1,1}) = 71.01\%$	Synergistic (both needed)	50%	50%
$\pi_{0011}(\mathcal{O}_{1,1}) = \text{pr}(G = 0011 \mid \mathcal{O}_{1,1}) = 14.16\%$	Smoking-only	100%	0%
$\pi_{0101}(\mathcal{O}_{1,1}) = \text{pr}(G = 0101 \mid \mathcal{O}_{1,1}) = 7.84\%$	Asbestos-only	0%	100%
$\pi_{0111}(\mathcal{O}_{1,1}) = \text{pr}(G = 0111 \mid \mathcal{O}_{1,1}) = 4.35\%$	Parallel (each sufficient)	50%	50%
$\pi_{1111}(\mathcal{O}_{1,1}) = \text{pr}(G = 1111 \mid \mathcal{O}_{1,1}) = 2.64\%$	Doomed (always occurs)	0%	0%
$\pi_{rstu}(\mathcal{O}_{1,1}) = \text{pr}(G = rstu \mid \mathcal{O}_{1,1}) = 0\%$	other groups	NA	NA

Smoking Responsibility: $71.01\% \times 50\% + 14.16\% \times 100\% + 4.35\% \times 50\% = \mathbf{51.84\%}$

Asbestos Responsibility: $71.01\% \times 50\% + 7.84\% \times 100\% + 4.35\% \times 50\% = \mathbf{45.52\%}$

Other Factors: $100\% - 51.84\% - 45.52\% = 2.64\%$

Expressions for Posterior Probabilities

- Standard causal inference identifies *marginal* distributions, e.g.,

$$\delta_{z,m} = \text{pr}(Y_{z,m} = 1) = \text{pr}(Y = 1 \mid Z = z, M = m)$$

- Identifying $\pi_{rstu}(\mathcal{O})$ therefore requires identifying the full joint π_{rstu},

$$\pi_{rstu} = \text{pr}(Y_{0,0} = r,\ Y_{0,1} = s,\ Y_{1,0} = t,\ Y_{1,1} = u)$$

Why the Joint Distribution?

The posterior probability $\pi_{rstu}(\mathcal{O})$ of causal type $G = rstu$ given observed evidence $\mathcal{O} = (Z = z, M = m, Y = y)$ is:

$$\pi_{rstu}(\mathcal{O}) = \mathbb{I}(Y=1)\,\pi_{1stu}/\delta_{0,0} + \mathbb{I}(Y=0)\,\pi_{0stu}/(1-\delta_{0,0}), \quad \mathcal{O} = (Z=0, M=0, Y)$$

$$\pi_{rstu}(\mathcal{O}) = \mathbb{I}(Y=1)\,\pi_{r1tu}/\delta_{0,1} + \mathbb{I}(Y=0)\,\pi_{r0tu}/(1-\delta_{0,1}), \quad \mathcal{O} = (Z=0, M=1, Y)$$

$$\pi_{rstu}(\mathcal{O}) = \mathbb{I}(Y=1)\,\pi_{rs1u}/\delta_{1,0} + \mathbb{I}(Y=0)\,\pi_{rs0u}/(1-\delta_{1,0}), \quad \mathcal{O} = (Z=1, M=0, Y)$$

$$\pi_{rstu}(\mathcal{O}) = \mathbb{I}(Y=1)\,\pi_{rst1}/\delta_{1,1} + \mathbb{I}(Y=0)\,\pi_{rst0}/(1-\delta_{1,1}), \quad \mathcal{O} = (Z=1, M=1, Y)$$

- This goes strictly beyond what randomization alone can provide

Causal Structure

$(Z=0, M=0, Y=0)$	$G=0000$	$G=0001$	$G=0010$	$G=0011$	$G=0100$	$G=0101$	$G=0110$	$G=0111$
$(Z=0, M=0, Y=1)$	$G=1000$	$G=1001$	$G=1010$	$G=1011$	$G=1100$	$G=1101$	$G=1110$	$G=1111$
$(Z=0, M=1, Y=0)$	$G=0000$	$G=0001$	$G=0010$	$G=0011$	$G=1000$	$G=1001$	$G=1010$	$G=1011$
$(Z=0, M=1, Y=1)$	$G=0100$	$G=0101$	$G=0110$	$G=0111$	$G=1100$	$G=1101$	$G=1110$	$G=1111$
$(Z=1, M=0, Y=0)$	$G=0000$	$G=0001$	$G=0100$	$G=0101$	$G=1000$	$G=1001$	$G=1100$	$G=1101$
$(Z=1, M=0, Y=1)$	$G=0010$	$G=0011$	$G=0110$	$G=0111$	$G=1010$	$G=1011$	$G=1110$	$G=1111$
$(Z=1, M=1, Y=0)$	$G=0000$	$G=0100$	$G=1000$	$G=1100$	$G=0010$	$G=0110$	$G=1010$	$G=1110$
$(Z=1, M=1, Y=1)$	$G=0001$	$G=0101$	$G=1001$	$G=1101$	$G=0011$	$G=0111$	$G=1011$	$G=1111$

- Each color represents one type $G = rstu$; Each observed probability is a mixture of 8 type probabilities.
- Each type $G = rstu$ appears in exactly four groups. For example, $G = 0000$ appears in all four groups.
- **Problem:** 4 observed probabilities, but 16 unknown type probabilities; since they sum to one, we have 15 independent parameters.

$\Rightarrow$ Not identifiable without additional assumptions!

Outline

Partial Identification and Identification

Monotonicity Assumption

Monotonicity

For any individual, increasing either exposure weakly increases the outcome.

$$Y_{0,0} \leq Y_{0,1} \leq Y_{1,1} \quad \text{and} \quad Y_{0,0} \leq Y_{1,0} \leq Y_{1,1}$$

- This assumption is widely adopted in the epidemiological literature, implying that adding exposures cannot reduce the outcome.
- **For smoking and asbestos:** a very reasonable assumption — we do not believe smoking or asbestos protects against lung cancer.
- Reduces from 16 types to **6 biologically plausible types**.
- The relationship between $Y_{0,1}$ and $Y_{1,0}$ remains unrestricted, allowing for heterogeneous causal effects.

Monotonicity Assumption

Type	Logic Expression	Interpretation
0000	$\emptyset$	Lung cancer never occurs (**immune**)
0001	$\{Z=1\} \cap \{M=1\}$	Lung cancer occurs if and only if smoking with asbestos exposure is present (**synergistic**)
0011	$\{Z=1\}$	Lung cancer occurs if and only if smoking is present (**smoking**)
0101	$\{M=1\}$	Lung cancer occurs if and only if asbestos exposure is present (**asbestos**)
0111	$\{Z=1\} \cup \{M=1\}$	Lung cancer occurs if and only if either smoking or asbestos exposure is present (**parallel**)
1111	Ω	Lung cancer always occurs (**doomed**)

Result: Monotonicity $Y_{0,0} \leq Y_{0,1} \leq Y_{1,1}$, $Y_{0,0} \leq Y_{1,0} \leq Y_{1,1}$ rules out 10 implausible types, leaves 6 biologically reasonable ones

With Monotonicity: Mixture of 6 Types

$(Z=0, M=0, Y=0)$	$G=0000$	$G=0001$	$G=0011$	$G=0101$	$G=0111$
$(Z=0, M=0, Y=1)$	$G=1111$				
$(Z=0, M=1, Y=0)$	$G=0000$	$G=0001$	$G=0011$		
$(Z=0, M=1, Y=1)$	$G=0101$	$G=0111$	$G=1111$		
$(Z=1, M=0, Y=0)$	$G=0000$	$G=0001$	$G=0101$		
$(Z=1, M=0, Y=1)$	$G=0011$	$G=0111$	$G=1111$		
$(Z=1, M=1, Y=0)$	$G=0000$				
$(Z=1, M=1, Y=1)$	$G=0001$	$G=0011$	$G=0101$	$G=0111$	$G=1111$

The Fundamental Problem

Under monotonicity, there are 6 latent types with probabilities summing to 1, leaving 5 free parameters. However, there are only 4 observable marginal probabilities; identification is still not achieved.

Sharp Bounds Under Monotonicity

Key Quantity

Let $\delta_{z,m} = \text{pr}(Y_{z,m} = 1) = \text{pr}(Y = 1 \mid Z = z, M = m)$. The only non-identifiable free parameter is $\nu = \text{pr}(Y_{0,1} = 1, Y_{1,0} = 1)$, with sharp bounds:

$$\nu^L = \max\{\delta_{0,0},\ \delta_{0,1} + \delta_{1,0} - \delta_{1,1}\} \le \nu \le \min\{\delta_{0,1},\ \delta_{1,0}\} = \nu^U$$

Type probabilities:

$$\pi_{0000} = 1 - \delta_{1,1},\ \pi_{1111} = \delta_{0,0}$$

$$\pi_{0001} = \delta_{1,1} - \delta_{0,1} - \delta_{1,0} + \nu \ge 0$$

$$\pi_{0011} = \delta_{1,0} - \nu$$

$$\pi_{0101} = \delta_{0,1} - \nu$$

$$\pi_{0111} = \nu - \delta_{0,0}$$

Synergistic effect:

Under monotonicity, $\pi_{0001} > 0$ is guaranteed when $\delta_{0,1} + \delta_{1,0} < \delta_{1,1}$.

Posterior bounds under $\mathcal{O}_{1,1}$:

since $\pi_{rst1}(\mathcal{O}_{1,1}) = \pi_{rst1}/\delta_{1,1}$,

$$\pi_{0001}(\mathcal{O}_{1,1}) \in \left[\frac{\delta_{1,1}-\delta_{0,1}-\delta_{1,0}+\nu^L}{\delta_{1,1}},\ \frac{\delta_{1,1}-\delta_{0,1}-\delta_{1,0}+\nu^U}{\delta_{1,1}}\right]$$

$$\pi_{0011}(\mathcal{O}_{1,1}) \in \left[\frac{\delta_{1,0}-\nu^U}{\delta_{1,1}},\ \frac{\delta_{1,0}-\nu^L}{\delta_{1,1}}\right]$$

$$\pi_{0101}(\mathcal{O}_{1,1}) \in \left[\frac{\delta_{0,1}-\nu^U}{\delta_{1,1}},\ \frac{\delta_{0,1}-\nu^L}{\delta_{1,1}}\right]$$

$$\pi_{0111}(\mathcal{O}_{1,1}) \in \left[\frac{\nu^L-\delta_{0,0}}{\delta_{1,1}},\ \frac{\nu^U-\delta_{0,0}}{\delta_{1,1}}\right]$$

$$\pi_{1111}(\mathcal{O}_{1,1}) = \frac{\delta_{0,0}}{\delta_{1,1}}$$

Point Identification via Maximum Entropy

Darroch & Borkent (1994)

Choose ν to maximize:

$$H = -\sum_{g \in \{0001,0011,0101,0111\}} \pi_g \log \pi_g$$

subject to constraints from $\{\delta_{z,m}\}$.

The maximum entropy solution is:

$$\nu^* = \delta_{0,0} + \frac{(\delta_{0,1} - \delta_{0,0})(\delta_{1,0} - \delta_{0,0})}{\delta_{1,1} - \delta_{0,0}}$$

Selects the least informative distribution consistent with the observed data, i.e., the distribution with maximum entropy.

Posterior probabilities under $\mathcal{O}_{1,1}$:

Substituting ν^* into $\pi_{rst1}(\mathcal{O}_{1,1})$:

$$\pi^*_{0001}(\mathcal{O}_{1,1}) = \frac{\delta_{1,1} - \delta_{0,1} - \delta_{1,0} + \nu^*}{\delta_{1,1}}$$

$$\pi^*_{0011}(\mathcal{O}_{1,1}) = \frac{\delta_{1,0} - \nu^*}{\delta_{1,1}}$$

$$\pi^*_{0101}(\mathcal{O}_{1,1}) = \frac{\delta_{0,1} - \nu^*}{\delta_{1,1}}$$

$$\pi^*_{0111}(\mathcal{O}_{1,1}) = \frac{\nu^* - \delta_{0,0}}{\delta_{1,1}}$$

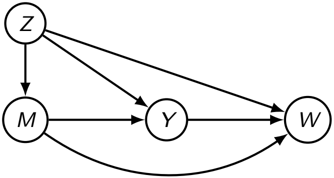
$$\pi^*_{1111}(\mathcal{O}_{1,1}) = \frac{\delta_{0,0}}{\delta_{1,1}}$$

Darroch, John N. and Borkent, M. (1994). Synergism, attributable risk and interaction for two binary exposure factors. *Biometrika*, **81**(2), 259–270.

Post-Treatment Proxy for Latent Type I

Key idea: We introduce a secondary outcome W as a proxy for the latent causal type G.

Unlike instrumental variables or proximal causal inference, auxiliary variables are pre-treatment; here W is *post-treatment*.



e.g., a biomarker measured after exposure

Key Observation

For the observed evidence $\mathcal{O} = (Z = 1, M = 1, Y = 1)$, the conditional density of W is a finite mixture:

$$f(W \mid \mathcal{O}) = \sum_{r,s,t} \pi_{rst1}(\mathcal{O}) \cdot f(W \mid G = rst1, \mathcal{O})$$

Identifiability of this mixture density, together with its corresponding weights, yields identifiability of π_{rstu} and $\pi_{rstu}(\mathcal{O})$.

Post-Treatment Proxy for Latent Type II

Gaussian Mixture Setup: Assume that conditional on the latent type $G = rstu$ and exposure $(Z = z, M = m)$, the proxy W follows a normal distribution:

$$W \mid G = rstu,\ Z = z,\ M = m\ \sim\ \mathcal{N}(\mu_{z,m,rstu},\ \sigma^2_{z,m,rstu})$$

with type-specific mean $\mu_{z,m,rstu}$ and variance $\sigma^2_{z,m,rstu}$. This gives rise to the mixture:

$$f(W \mid \mathcal{O}) = \sum_{r,s,t,u} \pi_{rstu}(\mathcal{O}) \cdot \mathcal{N}(\mu_{z,m,rstu},\ \sigma^2_{z,m,rstu})$$

$(Z=0, M=0, Y=0)$	$G=0000$	$G=0001$	$G=0010$	$G=0011$	$G=0100$	$G=0101$	$G=0110$	$G=0111$
$(Z=0, M=0, Y=1)$	$G=1000$	$G=1001$	$G=1010$	$G=1011$	$G=1100$	$G=1101$	$G=1110$	$G=1111$
$(Z=0, M=1, Y=0)$	$G=0000$	$G=0001$	$G=0010$	$G=0011$	$G=1000$	$G=1001$	$G=1010$	$G=1011$
$(Z=0, M=1, Y=1)$	$G=0100$	$G=0101$	$G=0110$	$G=0111$	$G=1100$	$G=1101$	$G=1110$	$G=1111$
$(Z=1, M=0, Y=0)$	$G=0000$	$G=0001$	$G=0100$	$G=0101$	$G=1000$	$G=1001$	$G=1100$	$G=1101$
$(Z=1, M=0, Y=1)$	$G=0010$	$G=0011$	$G=0110$	$G=0111$	$G=1010$	$G=1011$	$G=1110$	$G=1111$
$(Z=1, M=1, Y=0)$	$G=0000$	$G=0100$	$G=1000$	$G=1100$	$G=0010$	$G=0110$	$G=1010$	$G=1110$
$(Z=1, M=1, Y=1)$	$G=0001$	$G=0101$	$G=1001$	$G=1101$	$G=0011$	$G=0111$	$G=1011$	$G=1111$

A Classical Result: Gaussian Mixtures Are Identifiable Up to Label

It is well known (Titterington et al., 1985) that a finite Gaussian mixture is identifiable up to label permutation. That is, from $f(W \mid \mathcal{O})$ alone, we can recover:

$$\{\pi_{rstu}(\mathcal{O}),\ \mu_{z,m,rstu},\ \sigma^2_{z,m,rstu}\} \quad \text{for all } r, s, t, u$$

up to relabeling of components.

- The normality assumption on $f(W \mid Z = z, M = m, G = rstu)$ is used only to invoke identifiability of Gaussian mixtures (Titterington et al., 1985); it can be replaced by other distributions, e.g., Gamma or Binomial.
- The mixing weights $\pi_{rstu}(\mathcal{O})$ are *exactly* the causal attribution quantities we care about; **no parametric restrictions** are imposed on π_{rstu} or $\pi_{rstu}(\mathcal{O})$.
- **We just need to label the components.** Note that $G = rstu$ each appears as a shared component across only four mixtures, providing natural anchors for labeling.

Post-Treatment Proxy for Latent Type IV

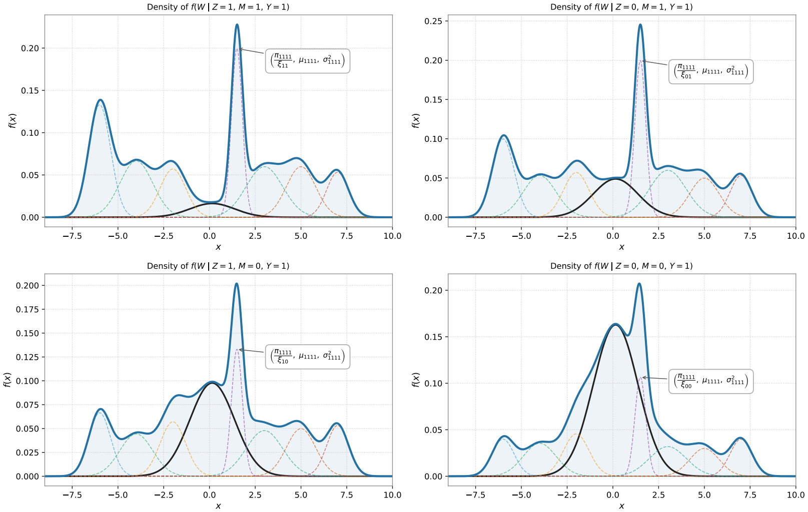


Figure: Conditional densities $f(W \mid Z, M, Y = 1)$ under four exposure combinations. Each mixture contains 8 Gaussian components.

Identification Without Monotonicity I

Assumption (Distinguishability)

For any $z, m \in \{0,1\}$, at least one of the following holds: (i) all means $\mu_{z,m,g} = \mu_g$ are distinct (exclusion restriction); (ii) all variances $\sigma^2_{z,m,g} = \sigma^2_g$ are distinct (exclusion restriction); (iii) all π_g are distinct (uniqueness).

Theorem 1 (Without monotonicity): Under ignorability, normality, and distinguishability, the causal attribution quantities π_{rstu} and $\pi_{rstu}(\mathcal{O})$ are identifiable.

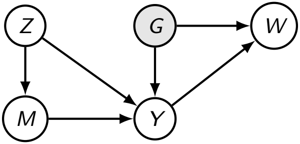


Figure: Illustration for exclusion restriction.

$$f(W \mid \mathcal{O}) = \sum_{r,s,t,u} \pi_{rstu}(\mathcal{O}) \cdot \mathcal{N}(\mu_{z,m,rstu},\ \sigma^2_{z,m,rstu}) = \sum_{r,s,t,u} \pi_{rstu}(\mathcal{O}) \cdot \mathcal{N}(\mu_{rstu},\ \sigma^2_{rstu})$$

Condition (i): Distinct means — μ_{rstu} are distinct across types.

A finite Gaussian mixture is identifiable only *up to permutation of labels*: without additional information, we cannot tell which component corresponds to which latent type.

$f(W \mid Z=0, M=0, Y=0)$								
$f(W \mid Z=0, M=0, Y=1)$								
$f(W \mid Z=0, M=1, Y=0)$								
$f(W \mid Z=0, M=1, Y=1)$								
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$f(W \mid Z=1, M=1, Y=0)$								
$f(W \mid Z=1, M=1, Y=1)$								

Key observation: for any $y_{0,0}, y_{0,1}, y_{1,0}, y_{1,1} \in \{0,1\}$, the four densities $f(W \mid Z=z, M=m, Y=y_{z,m})$ over $(z,m) \in \{0,1\}^2$ share exactly one common component—e.g., $y_{z,m} \equiv 0$ yields the pink component $(G=0000)$.

Identification Without Monotonicity III

$f(W \mid Z=0, M=0, Y=0)$	μ_{0000}	μ_{0001}	μ_{0010}	μ_{0011}	μ_{0100}	μ_{0101}	μ_{0110}	μ_{0111}
$f(W \mid Z=0, M=0, Y=1)$	μ_{1000}	μ_{1001}	μ_{1010}	μ_{1011}	μ_{1100}	μ_{1101}	μ_{1110}	μ_{1111}
$f(W \mid Z=0, M=1, Y=0)$	μ_{0000}	μ_{0001}	μ_{0010}	μ_{0011}	μ_{1000}	μ_{1001}	μ_{1010}	μ_{1011}
$f(W \mid Z=0, M=1, Y=1)$	μ_{0100}	μ_{0101}	μ_{0110}	μ_{0111}	μ_{1100}	μ_{1101}	μ_{1110}	μ_{1111}
$f(W \mid Z=1, M=0, Y=0)$	μ_{0000}	μ_{0001}	μ_{0100}	μ_{0101}	μ_{1000}	μ_{1001}	μ_{1100}	μ_{1101}
$f(W \mid Z=1, M=0, Y=1)$	μ_{0010}	μ_{0011}	μ_{0110}	μ_{0111}	μ_{1010}	μ_{1011}	μ_{1110}	μ_{1111}
$f(W \mid Z=1, M=1, Y=0)$	μ_{0000}	μ_{0100}	μ_{1000}	μ_{1100}	μ_{0010}	μ_{0110}	μ_{1010}	μ_{1110}
$f(W \mid Z=1, M=1, Y=1)$	μ_{0001}	μ_{0101}	μ_{1001}	μ_{1101}	μ_{0011}	μ_{0111}	μ_{1011}	μ_{1111}

Key: By matching this shared mean across strata, we uniquely assign μ_{0000} to it—and similarly label all other components.

$\Rightarrow$ Labels assigned ✓

$\Rightarrow$ Weights $\pi_{rstu}/\xi_{z,m}$ identified ✓

$\Rightarrow$ $\pi_{rstu}(\mathcal{O})$ identified ✓

Identification with Monotonicity I

Assumption (Normality)

$W \mid (Z = z, M = m, G = g) \sim \mathcal{N}(\mu_{z,m,g},\ \sigma^2_{z,m,g})$, with distinct parameter pairs across latent types. *(identifiability for Gaussian mixture model)*

Theorem 2 (With monotonicity): Under ignorability, monotonicity, and normality, the causal attribution quantities π_{rstu} and $\pi_{rstu}(\mathcal{O})$ are identifiable.

$(Z = 0, M = 0, Y = 0)$	$G = 0000$	$G = 0001$	$G = 0011$	$G = 0101$	$G = 0111$
$(Z = 0, M = 0, Y = 1)$	$G = 1111$				
$(Z = 0, M = 1, Y = 0)$	$G = 0000$	$G = 0001$	$G = 0011$		
$(Z = 0, M = 1, Y = 1)$	$G = 0101$	$G = 0111$	$G = 1111$		
$(Z = 1, M = 0, Y = 0)$	$G = 0000$	$G = 0001$	$G = 0101$		
$(Z = 1, M = 0, Y = 1)$	$G = 0011$	$G = 0111$	$G = 1111$		
$(Z = 1, M = 1, Y = 0)$	$G = 0000$				
$(Z = 1, M = 1, Y = 1)$	$G = 0001$	$G = 0011$	$G = 0101$	$G = 0111$	$G = 1111$

Beyond Identification

	Nonparametric Bounds	**Maximum Entropy**	**Proposed 1**	**Proposed 2**
Identifiability	Partial (interval)	Point estimate*	**Full identification**	**Full identification**
Assumption	Monotonicity	Monotonicity	Monotonicity + Normality	Normality + Distinguishability
Secondary outcome	No	No	**Yes**	**Yes**

*Maximum entropy yields a point estimate within the bounds but does not guarantee convergence to the true parameter.

- Under the assumptions of monotonicity, ignorability, and normality, Theorem 1 ensures the identifiability of $\mathrm{pr}(G = rst1 \mid Z = 1, M = 1, W)$.
- Without the monotonicity assumption, Theorem 2 guarantees the identifiability of $\mathrm{pr}(G = rst1 \mid Z = 1, M = 1, W)$ under ignorability and class-separability conditions.

Outline

Empirical Application

Context: We examine causal attribution in the Telemark study of smoking and asbestos exposure under the monotonicity assumption.

Population: 21,319 Norwegian males surveyed during 1982–1983.

- $Z = 1$ (Smoker), $Z = 0$ (Non-smoker)
- $M = 1$ (Asbestos exposed), $M = 0$ (Unexposed)
- $Y = 1$ (Lung cancer), $Y = 0$ (No cancer)
- W (BMI, secondary outcome)

Goal: Use the proposed framework to attribute lung cancer cases to exposures.

	No asbestos exposure ($M = 0$)	Asbestos exposure ($M = 1$)
Non-smokers ($Z = 0$)	$\xi_{0,0} = \frac{6}{5057} \approx 0.12\%$	$\xi_{0,1} = \frac{5}{749} \approx 0.67\%$
Smokers ($Z = 1$)	$\xi_{1,0} = \frac{118}{12383} \approx 0.95\%$	$\xi_{1,1} = \frac{141}{3130} \approx 4.51\%$

Table: Lung cancer incidence by exposure group

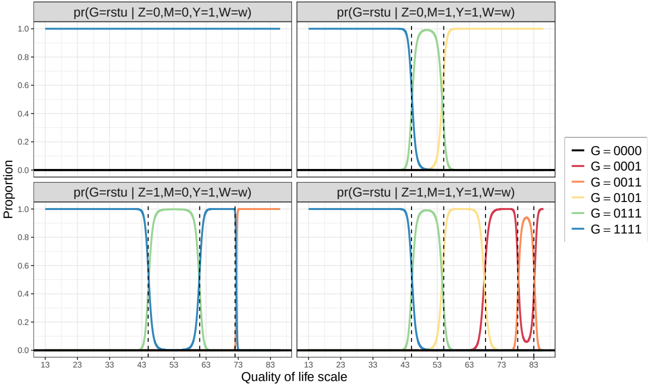
Method Comparison for $\mathcal{O}_{1,1}$

	Nonparametric Bounds	Maximum Entropy	Proposed Method
Identifiability	Partial (interval)	Point estimate*	**Full identification**
Assumption	Monotonicity only	Monotonicity only	Monotonicity + Normality
Uses secondary W	No	No	**Yes**
$\pi_{0001}(\mathcal{O}_{1,1})$	[66.66%, 78.85%]	68.97%	**71.01%**
$\pi_{0011}(\mathcal{O}_{1,1})$	[6.33%, 18.51%]	16.20%	14.16%
$\pi_{0101}(\mathcal{O}_{1,1})$	[0%, 12.19%]	9.88%	7.84%
$\pi_{1111}(\mathcal{O}_{1,1})$	2.64%	2.64%	2.64%

*Maximum entropy yields a point estimate within the bounds but does not guarantee convergence to the true parameter.

- All three methods agree: the **synergistic** class ($G = 0001$) dominates ($\approx$ 67–71%), suggesting co-occurrence is the primary cause rather than either exposure alone.
- The data show clear evidence of synergy: $\delta_{0,1} + \delta_{1,0} < \delta_{1,1}$.
- The proposed method **fully identifies** $\pi_{rstu}(\mathcal{O}_{1,1})$ by leveraging the Gaussian mixture structure of W.

Secondary Outcome Estimation



- For $(Z = 0, M = 0, Y = 1)$: $G = 0000$ dominates for all W, as individuals with no exposure are almost certainly background cases.
- For $(Z = 0, M = 1, Y = 1)$ and $(Z = 1, M = 0, Y = 1)$: the posterior concentrates on one type, switching sharply at a threshold value of W.
- For $(Z = 1, M = 1, Y = 1)$: multiple types compete, with $G = 0001$ and others alternating as W varies, reflecting more uncertainty than the other three scenarios.

Conclusion

Conclusion

1. **Causal attribution framework.** A two-layer framework was proposed, combining causal types with posterior probabilities to enable cause-specific attribution for two binary exposures.

2. **Partial identification.** Under monotonicity, nonparametric bounds and maximum entropy were shown to provide informative constraints on $\pi_{rstu}(\mathcal{O})$.

3. **Full identification.** Introducing a secondary outcome W under a Gaussian mixture model was shown to achieve full identification of $\pi_{rstu}(\mathcal{O})$.

4. **Application.** Applied to the Telemark study, $\approx$ 69–71% of lung cancer cases among dual-exposed individuals are attributable to the **synergistic interaction** of smoking and asbestos.

Thank You for Your Attention!

Questions?